

A SIMPLE CHARACTERIZATION OF CHAOS FOR WEIGHTED COMPOSITION C_0 -SEMIGROUPS ON LEBESGUE AND SOBOLEV SPACES

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ABSTRACT. We give a simple characterization of chaos for weighted composition C_0 -semigroups on $L^p_\rho(\Omega)$ for an open interval $\Omega \subseteq \mathbb{R}$. Moreover, we characterize chaos for these classes of C_0 -semigroups on the closed subspace $W_*^{1,p}(\Omega)$ of the Sobolev space $W^{1,p}(\Omega)$ for a bounded interval $\Omega \subset \mathbb{R}$. These characterizations simplify the characterization of chaos obtained in [3] for these classes of C_0 -semigroups.

1. INTRODUCTION

The purpose of this article is to give a simple characterization of chaos for certain weighted composition C_0 -semigroups on Lebesgue spaces and Sobolev spaces over open intervals. Recall that a C_0 -semigroup T on a separable Banach space X is called *chaotic* if T is hypercyclic, i.e. there is $x \in X$ such that $\{T(t)x; t \geq 0\}$ is dense in X , and if the set of periodic points, i.e. $\{x \in X; \exists t > 0 : T(t)x = x\}$, is dense in X .

The study of chaotic C_0 -semigroups has attracted the attention of many researchers. We refer the reader to Chapter 7 of the monograph by Grosse-Erdmann and Peris [9] and the references therein. Some recent papers in the topic are [1, 4, 5, 8, 14]. For $\Omega \subseteq \mathbb{R}$ open and a Borel measure μ on Ω admitting a strictly positive Lebesgue density ρ we consider C_0 -semigroups T on $L^p(\Omega, \mu)$, $1 \leq p < \infty$, of the form

$$T(t)f(x) = h_t(x)f(\varphi(t, x)),$$

where φ is the solution semiflow of an ordinary differential equation

$$\dot{x} = F(x)$$

in Ω and

$$h_t(x) = \exp\left(\int_0^t h(\varphi(s, x))ds\right)$$

with $h \in C(\Omega)$. Such C_0 -semigroups appear in a natural way when dealing with initial value problems for linear first order partial differential operators. While a characterization of chaos for such C_0 -semigroups was obtained for open $\Omega \subseteq \mathbb{R}^d$ for arbitrary dimension d in [10], evaluation of these conditions in concrete examples is sometimes rather involved. In contrast to general dimension the case $d = 1$ allows for a significantly simplified characterization, see [3]. However, this characterization of chaos still depends on the knowledge of the solution semiflow φ which might be difficult to determine in concrete examples.

In section 2 we give, under mild additional assumptions on F and h , a characterization of chaos which only depends on the ingredients F , h , and ρ , without referring to the semiflow φ .

In section 3 we use this result to obtain a similarly simple characterization of chaos for the above kind of C_0 -semigroups acting on the closed subspace

$$W_*^{1,p}[a, b] = \{f \in W^{1,p}[a, b]; f(a) = 0\}$$

of the Sobolev spaces $W^{1,p}[a, b]$, where $(a, b) \subseteq \mathbb{R}$ is a bounded interval. It was shown in [3] that such C_0 -semigroups cannot be hypercyclic, a fortiori chaotic, on the whole Sobolev space $W^{1,p}(a, b)$.

In order to illustrate our results, several examples are considered.

2. CHAOTIC WEIGHTED COMPOSITION C_0 -SEMIGROUPS ON LEBESGUE SPACES

Let $\Omega \subseteq \mathbb{R}$ be open and let $F : \Omega \rightarrow \mathbb{R}$ be a C^1 -function. Hence, for every $x_0 \in \Omega$ there is a unique solution $\varphi(\cdot, x_0)$ of the initial value problem

$$\dot{x} = F(x), \quad x(0) = x_0.$$

Denoting its maximal domain of definition by $J(x_0)$ it is well-known that $J(x_0)$ is an open interval containing 0. We make the general assumption that Ω is *forward invariant under F* , i.e. $[0, \infty) \subset J(x_0)$ for every $x_0 \in \Omega$, that is $\varphi : [0, \infty) \rightarrow \Omega$. This is true, for example, if $\Omega = (a, b)$ is a bounded interval and if F can be extended to a C^1 -function defined on a neighborhood of $[a, b]$ such that $F(a) \geq 0$ and $F(b) \leq 0$ (cf. [2, Corollary 16.10]).

From the uniqueness of the solution it follows that $\varphi(t, \cdot)$ is injective for every $t \geq 0$ and $\varphi(t + s, x) = \varphi(t, \varphi(s, x))$ for all $x \in \Omega$ and $s, t \in J(x)$ with $s + t \in J(x)$. Moreover, for every $t \geq 0$ the set $\varphi(t, \Omega)$ is open and for $x \in \varphi(t, \Omega)$ we have $[-t, \infty) \subset J(x)$ as well as $\varphi(-s, x) = \varphi(s, \cdot)^{-1}(x)$ for all $s \in [0, t]$. Since F is a C^1 -function it is well-known that the same is true for $\varphi(t, \cdot)$ on Ω and $\varphi(-t, \cdot)$ on $\varphi(t, \Omega)$ for every $t \geq 0$.

Moreover, let $h \in C(\Omega)$ and define for $t \geq 0$

$$h_t : \Omega \rightarrow \mathbb{C}, \quad h_t(x) = \exp\left(\int_0^t h(\varphi(s, x)) ds\right).$$

For $1 \leq p < \infty$ and a measurable function $\rho : \Omega \rightarrow (0, \infty)$ let $L_\rho^p(\Omega)$ be as usual the Lebesgue space of p -integrable functions with respect to the Borel measure $\rho d\lambda$, where λ denotes Lebesgue measure. If Ω is forward invariant under F the operators

$$T(t) : L_\rho^p(\Omega) \rightarrow L_\rho^p(\Omega), \quad (T(t)f)(x) := h_t(x)f(\varphi(t, x)) \quad (t \geq 0)$$

are well-defined continuous linear operators defining a C_0 -semigroup $T_{F,h}$ on $L_\rho^p(\Omega)$ if ρ is p -admissible for F and h , i.e. if there are constants $M \geq 1$, $\omega \in \mathbb{R}$ with

$$\forall t \geq 0, x \in \Omega : |h_t(x)|^p \rho(x) \leq M e^{\omega t} \rho(\varphi(t, x)) \exp\left(\int_0^t F'(\varphi(s, x)) ds\right),$$

(see [3]). Because $|h_t(x)|^p = \exp(p \int_0^t \operatorname{Re} h(\varphi(s, x)) ds)$ it follows that $\rho = 1$ is p -admissible for any p if $\operatorname{Re} h$ is bounded above and F' is bounded below, i.e. in this case the above operators define a C_0 -semigroup $T_{F,h}$ on the standard Lebesgue spaces $L^p(\Omega)$. Under mild additional assumptions on F and h the generator of this C_0 -semigroup is given by the first order differential operator $Af = Ff' + hf$ on a suitable subspace of $L^p(\Omega)$ (see [3, Theorem 15]).

In [3, Theorem 6 and Proposition 9] it is characterized when the C_0 -semigroup $T_{F,h}$ is chaotic on $L_\rho^p(\Omega)$. However, this characterization depends on a more or less explicit knowledge of the semiflow φ .

Our aim is to prove the following characterization of chaos for $T_{F,h}$ on $L_\rho^p(\Omega)$ valid under mild additional assumptions on F and h and which is given solely in terms of F , h , and ρ . Throughout this article, we use the following common abbreviation $\{F = 0\} := \{x \in \Omega; F(x) = 0\}$.

Theorem 1. *For $1 \leq p < \infty$ let $\Omega \subset \mathbb{R}$ be an open interval which is forward invariant under $F \in C^1(\Omega)$, and let $h \in C(\Omega)$ be such that F' and $\operatorname{Re} h$ are bounded and*

- a) There is $\gamma \in \mathbb{R}$ such that $h(x) = \gamma$ for all $x \in \{F = 0\}$.
b) With $\alpha := \inf \Omega$ and $\omega := \sup \Omega$ the function

$$\Omega \rightarrow \mathbb{C}, y \mapsto \frac{\operatorname{Im} h(y)}{F(y)}$$

belongs to $L^1((\alpha, \beta))$ for all $\beta \in \Omega$ or to $L^1((\beta, \omega))$ for all $\beta \in \Omega$.

Then for every ρ which is p -admissible for F and h the following are equivalent.

- i) $T_{F,h}$ is chaotic in $L^p_\rho(\Omega)$.
ii) $\lambda(\{F = 0\}) = 0$ and for every connected component C of $\Omega \setminus \{F = 0\}$

$$\int_C \exp(-p \int_x^w \frac{\operatorname{Re} h(y)}{F(y)} dy) \rho(w) d\lambda(w) < \infty$$

for some/all $x \in C$.

In order to prove Theorem 1, we define for $x \in \Omega$, $p \geq 1$, and $t \geq 0$

$$\begin{aligned} \rho_{t,p}(x) &= \chi_{\varphi(t,\Omega)}(x) |h_t(\varphi(-t, x))|^p \exp\left(\int_0^{-t} F'(\varphi(s, x)) ds\right) \rho(\varphi(-t, x)) \\ &= \chi_{\varphi(t,\Omega)}(x) \exp\left(p \int_0^t \operatorname{Re} h(\varphi(s, \varphi(-t, x))) ds\right) \exp\left(\int_0^{-t} F'(\varphi(s, x)) ds\right) \rho(\varphi(-t, x)) \end{aligned}$$

as well as

$$\begin{aligned} \rho_{-t,p}(x) &= |h_t(x)|^{-p} \exp\left(\int_0^t F'(\varphi(s, x)) ds\right) \rho(\varphi(t, x)) \\ &= \exp\left(-p \int_0^t \operatorname{Re} h(\varphi(s, x)) ds\right) \exp\left(\int_0^t F'(\varphi(s, x)) ds\right) \rho(\varphi(t, x)). \end{aligned}$$

Then $\rho_{0,p} = \rho$, $\rho_{t,p} \geq 0$ for every $t \in \mathbb{R}$, and for fixed $x \in \Omega$ the mapping $t \mapsto \rho_{t,p}(x)$ is Lebesgue measurable. Moreover, it follows that

$$\begin{aligned} \rho_{-(t+s),p}(x) &= \exp\left(-p \int_0^{t+s} \operatorname{Re} h(\varphi(r, x)) dr\right) \\ &\quad \cdot \exp\left(\int_0^{t+s} F'(\varphi(r, x)) dr\right) \rho(\varphi(t, \varphi(s, x))) \\ &= \exp\left(\int_0^s F'(\varphi(r, x)) - p \operatorname{Re} h(\varphi(r, x)) dr\right) \\ (1) \quad &\quad \cdot \exp\left(-p \int_0^t \operatorname{Re} h(\varphi(r, \varphi(s, x))) dr\right) \\ &\quad \cdot \exp\left(\int_0^t F'(\varphi(r, \varphi(s, x))) dr\right) \rho(\varphi(t, \varphi(s, x))) \\ &= \exp\left(\int_0^s F'(\varphi(r, x)) - p \operatorname{Re} h(\varphi(r, x)) dr\right) \rho_{-t,p}(\varphi(s, x)) \end{aligned}$$

and analogously

$$\begin{aligned} \rho_{(t+s),p}(x) &= \chi_{\varphi(t+s,\Omega)}(x) \exp\left(p \int_{-(t+s)}^0 \operatorname{Re} h(\varphi(r, x)) - \frac{1}{p} F'(\varphi(r, x)) dr\right) \\ (2) \quad &\quad \cdot \rho(\varphi(-(t+s), x)) \\ &= \chi_{\varphi(s,\Omega)}(x) \exp\left(p \int_{-s}^0 \operatorname{Re} h(\varphi(r, x)) - \frac{1}{p} F'(\varphi(r, x)) dr\right) \rho_{t,p}(\varphi(-s, x)). \end{aligned}$$

The following lemma will be used in the proof of the first auxiliary result. We cite it for the reader's convenience. For a proof see [11, Lemma 7].

Lemma 2. *Let $\Omega \subseteq \mathbb{R}$ be open, let $F \in C^1(\Omega)$ be such that Ω is forward invariant under F , and let $h \in C(\Omega)$ be real valued. Moreover, for fixed $1 \leq p < \infty$ let ρ be p -admissible for F and h . For $[a, b] \subset \Omega \setminus \{F = 0\}$ set $\alpha := a$ and $\beta := b$ if $F|_{[a, b]} > 0$, respectively $\alpha := b$ and $\beta := a$ if $F|_{[a, b]} < 0$. Then there is a constant $C > 0$ such that*

$$\forall x \in [a, b] : \frac{1}{C} \leq \rho(x) \leq C$$

as well as

$$\forall t \in \mathbb{R}, x \in [a, b] : \frac{1}{C} \rho_{t,p}(\alpha) \leq \rho_{t,p}(x) \leq C \rho_{t,p}(\beta).$$

Lemma 3. *Let $\Omega \subseteq \mathbb{R}$ be open and forward invariant under $F \in C^1(\Omega)$, let $h \in C(\Omega)$ be such that F' and $\text{Re } h$ are bounded. Moreover, let ρ be p -admissible for F and h , $1 \leq p < \infty$. Then the following are equivalent.*

- i) *For all $x \in \Omega \setminus \{F = 0\}$ there is $t_0 > 0$ such that $\sum_{k \in \mathbb{Z}} \rho_{kt_0,p}(x) < \infty$.*
- ii) *For all $x \in \Omega \setminus \{F = 0\} : \int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) < \infty$.*
- iii) *For all $x \in \Omega \setminus \{F = 0\}$ and $t_0 > 0 : \sum_{k \in \mathbb{Z}} \rho_{kt_0,p}(x) < \infty$.*

Proof. In order to show that i) implies ii) fix $x \in \Omega \setminus \{F = 0\}$ and choose $t_0 > 0$ according to i) for x . We distinguish two cases. If x belongs to $\cap_{t \geq 0} \varphi(t, \Omega)$ it follows by equation (2) and the boundedness of $\text{Re } h$ and F'

$$\begin{aligned} & \int_{[0, \infty)} \rho_{t,p}(x) d\lambda(t) = \sum_{k=0}^{\infty} \int_{[0, t_0]} \rho_{kt_0+s,p}(x) d\lambda(s) \\ &= \sum_{k=0}^{\infty} \int_{[0, t_0]} \chi_{\varphi(s, \Omega)}(x) \exp(p \int_{-s}^0 \text{Re } h(\varphi(r, x)) - \frac{1}{p} F'(\varphi(r, x)) dr) \\ & \quad \rho_{kt_0,p}(\varphi(-s, x)) d\lambda(s) \\ &\leq C \sum_{k=0}^{\infty} \int_{[0, t_0]} \chi_{\varphi(s, \Omega)}(x) \rho_{kt_0,p}(\varphi(-s, x)) d\lambda(s) \\ &= C \sum_{k=0}^{\infty} \int_{[0, t_0]} \rho_{kt_0,p}(\varphi(-s, x)) d\lambda(s) \\ &\leq \begin{cases} \tilde{C} \sum_{k=0}^{\infty} \rho_{kt_0,p}(x) & , F(x) > 0 \\ \tilde{C} \sum_{k=0}^{\infty} \rho_{kt_0,p}(\varphi(-t_0, x)) & , F(x) < 0, \end{cases} \end{aligned}$$

where C and \tilde{C} depend on t_0 and where in the last step we used lemma 2 for F and $\text{Re } h$. Since by equation (2) together with the boundedness of F' and $\text{Re } h$ we also have with suitable $D > 0$ that for all $k \geq 0$

$$\rho_{kt_0,p}(\varphi(-t_0, x)) \leq D \rho_{(k+1)t_0,p}(x),$$

the above shows the existence of $\hat{C} > 0$ such that

$$\int_{[0, \infty)} \rho_{t,p}(x) d\lambda(t) \leq \hat{C} \sum_{k=0}^{\infty} \rho_{kt_0,p}(x).$$

If x does not belong to $\cap_{t \geq 0} \varphi(t, \Omega)$ then $\int_{[0, \infty)} \rho_{t,p}(x) d\lambda(t) = \int_{[0, r]} \rho_{t,p}(x) d\lambda(t)$ for some $r > 0$. Combining lemma 2 for F and $\text{Re } h$ with equation (2), the boundedness

of F' and $\operatorname{Re} h$ gives for suitable $C > 0$

$$\begin{aligned} & \int_{[0,\infty)} \rho_{t,p}(x) d\lambda(t) \\ &= \int_{[0,r]} \chi_{\varphi(t,\Omega)}(x) \exp\left(p \int_{-t}^0 \operatorname{Re} h(\varphi(s,x)) - \frac{1}{p} F'(\varphi(s,x)) ds\right) \rho_{0,p}(\varphi(-t,x)) d\lambda(t) \\ &\leq C \int_{[0,r]} \chi_{\varphi(t,\Omega)}(x) \rho_{0,p}(\varphi(-t,x)) d\lambda(t) < \infty. \end{aligned}$$

Thus, if i) holds then $\int_{[0,\infty)} \rho_{t,p}(x) d\lambda(t) < \infty$ for all $x \in \Omega \setminus \{F = 0\}$.

Moreover, by equation (1) we obtain for every $x \in \Omega \setminus \{F = 0\}$ together with the boundedness of F' and $\operatorname{Re} h$

$$\begin{aligned} & \int_{(-\infty,0]} \rho_{t,p}(x) d\lambda(t) = \sum_{k=0}^{\infty} \int_{[-t_0,0]} \rho_{-(kt_0-s),p}(x) d\lambda(s) \\ &= \sum_{k=0}^{\infty} \int_{[-t_0,0]} \exp\left(\int_0^{-s} F'(\varphi(r,x)) - p \operatorname{Re} h(\varphi(r,x))\right) \rho_{-kt_0,p}(\varphi(-s,x)) d\lambda(s) \\ &\leq C \sum_{k=0}^{\infty} \int_{[-t_0,0]} \rho_{-kt_0,p}(\varphi(-s,x)) d\lambda(s) \\ &\leq \begin{cases} \tilde{C} \sum_{k=0}^{\infty} \rho_{-kt_0,p}(\varphi(t_0,x)) & , F(x) > 0 \\ \tilde{C} \sum_{k=0}^{\infty} \rho_{-kt_0,p}(x) & , F(x) < 0, \end{cases} \end{aligned}$$

where C and \tilde{C} again depend on t_0 and where in the last step we again used lemma 2 for F and $\operatorname{Re} h$. Equation (1) and the fact that F' and $\operatorname{Re} h$ are bounded yield the existence of $D > 0$ such that for all $k \geq 0$

$$\rho_{-kt_0,p}(\varphi(t_0,x)) \leq D \rho_{-(k+1)t_0,p}(x).$$

So the above gives

$$\int_{(-\infty,0]} \rho_{t,p}(x) d\lambda(t) \leq \hat{C} \sum_{k=0}^{\infty} \rho_{-kt_0,p}(x)$$

for some $\hat{C} > 0$. Hence, i) implies ii).

In order to show that ii) implies iii) we fix $t_0 > 0$ and $x \in \Omega \setminus \{F = 0\}$ and distinguish again two cases. If x does not belong to $\cap_{t \geq 0} \varphi(t, \Omega)$ there is $t_1 > 0$ such that $\rho_{t,p}(x) = 0$ for all $t > t_1$. Therefore, $\sum_{k=0}^{\infty} \rho_{kt_0,p}(x) < \infty$.

In case of $x \in \cap_{t \geq 0} \varphi(t, \Omega)$ it follows from equation (2) together with the boundedness of F' and $\operatorname{Re} h$ that for some $C > 0$

$$\begin{aligned} & \int_{[0,\infty)} \rho_{t,p}(x) d\lambda(t) = \sum_{k=0}^{\infty} \int_{[0,t_0]} \rho_{kt_0+t,p}(x) d\lambda(t) \\ &= \sum_{k=0}^{\infty} \int_{[0,t_0]} \exp\left(p \int_{-t}^0 \operatorname{Re} h(\varphi(r,x)) - \frac{1}{p} F'(\varphi(r,x)) dr\right) \rho_{kt_0,p}(\varphi(-t,x)) d\lambda(t) \\ &\geq \sum_{k=0}^{\infty} C \int_{[0,t_0]} \rho_{kt_0,p}(\varphi(-t,x)) d\lambda(t) \\ &\geq \begin{cases} \tilde{C} \sum_{k=0}^{\infty} \rho_{kt_0,p}(x) & , F(x) < 0 \\ \tilde{C} \rho_{kt_0,p}(\varphi(-t_0,x)) & , F(x) < 0, \end{cases} \end{aligned}$$

where we used lemma 2 in the last step. By equation (2) and the boundedness of F' and $\operatorname{Re} h$ we have $\rho_{kt_0,p}(\varphi(-t_0,x)) \geq D \rho_{(k+1)t_0,p}(x)$ for suitable $D > 0$ such

that the above gives

$$(3) \quad \int_{[0, \infty)} \rho_{t,p}(x) d\lambda(t) \geq \hat{C}_1 \sum_{k=0}^{\infty} \rho_{kt_0,p}(x)$$

for some \hat{C}_1 .

Additionally, applying lemma 2 for F and $\text{Re } h$ we also obtain from the boundedness of F' and $\text{Re } h$ together with equation (1)

$$\begin{aligned} \int_{(-\infty, 0]} \rho_{t,p}(x) d\lambda(t) &= \sum_{k=0}^{\infty} \int_{[-t_0, 0]} \rho_{-(kt_0-t),p}(x) d\lambda(t) \\ &= \sum_{k=0}^{\infty} \int_{[-t_0, 0]} \exp\left(\int_0^{-t} F'(\varphi(r, x)) - p \text{Re } h(\varphi(r, x)) dr\right) \rho_{-kt_0,p}(\varphi(-t, x)) d\lambda(t) \\ &\geq C \sum_{k=0}^{\infty} \int_{[-t_0, 0]} \rho_{-kt_0,p}(\varphi(-t, x)) d\lambda(t) \\ &\geq \begin{cases} \tilde{C} \sum_{k=0}^{\infty} \rho_{-kt_0,p}(x) & , F(x) > 0 \\ \tilde{C} \sum_{k=0}^{\infty} \rho_{-kt_0,p}(\varphi(-t_0, x)) & , F(x) < 0 \end{cases} \\ &\geq \hat{C}_2 \sum_{k=0}^{\infty} \rho_{-kt_0,p}(x). \end{aligned}$$

Hence, together with (3), iii) follows from ii), and as iii) obviously implies i), the lemma is proved. \square

The applicability of the previous lemma depends on an explicit knowledge of φ . The next lemma shows that the integrals appearing in the previous result can be expressed in terms of F , h , and ρ .

Lemma 4. *Let $\Omega \subseteq \mathbb{R}$ be open and forward invariant under $F \in C^1(\Omega)$, $h \in C(\Omega)$ and let ρ be p -admissible for F and h , $1 \leq p < \infty$. Then for every $x \in \Omega \setminus \{F = 0\}$ we have*

$$\int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) = \frac{1}{|F(x)|} \int_{C(x)} \exp\left(p \int_w^x \frac{\text{Re } h(y)}{F(y)} dy\right) \rho(w) d\lambda(w),$$

where $C(x)$ denotes the connected component of $\Omega \setminus \{F = 0\}$ containing x .

Proof. Fix $x \in \Omega \setminus \{F = 0\}$ and let $C(x)$ be as in the lemma. Observe that $\varphi(t, x) \in C(x)$ for all $t \in J(x)$ and that $\varphi(J(x), x) = C(x)$, where $J(x)$ is the domain of the maximal solution $\varphi(\cdot, x)$ of the initial value problem $\dot{y} = F(y)$, $y(0) = x$. Obviously,

$$\int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) = \int_{[0, \infty)} \rho_{t,p}(x) d\lambda(t) + \int_{[0, \infty)} \rho_{-t,p}(x) d\lambda(x).$$

We set $C^+(x) = \{\varphi(t, x); t \geq 0\}$. Applying the Transformation Formula for Lebesgue integrals we obtain with equation (1)

$$\begin{aligned} &\int_{[0, \infty)} \rho_{-t,p}(x) d\lambda(t) \\ &= \int_{[0, \infty)} \exp\left(\int_0^t F'(\varphi(r, x)) - p \text{Re } h(\varphi(r, x)) dr\right) \rho(\varphi(t, x)) d\lambda(t) \\ &= \int_{[0, \infty)} \exp\left(\int_0^t \frac{F'(\varphi(r, x)) - p \text{Re } h(\varphi(r, x))}{F(\varphi(r, x))} \partial_1 \varphi(r, x) dr\right) \rho(\varphi(t, x)) d\lambda(t) \\ &= \int_{[0, \infty)} \exp\left(\int_x^{\varphi(t, x)} \frac{F'(y) - p \text{Re } h(y)}{F(y)} dy\right) \rho(\varphi(t, x)) d\lambda(t) \end{aligned}$$

$$\begin{aligned}
&= \int_{(0,\infty)} \frac{\exp(\int_x^{\varphi(t,x)} \frac{F'(y)-p\operatorname{Re} h(y)}{F(y)} dy)}{|F(\varphi(t,x))|} \rho(\varphi(t,x)) |\partial_1 \varphi(t,x)| d\lambda(t) \\
&= \int_{C^+(x)} \frac{\exp(\int_x^w \frac{F'(y)-p\operatorname{Re} h(y)}{F(y)} dy)}{|F(w)|} \rho(w) d\lambda(w) \\
&= \int_{C^+(x)} \frac{\exp(\int_w^x \frac{p\operatorname{Re} h(y)-F'(y)}{F(y)} dy)}{|F(w)|} \rho(w) d\lambda(w).
\end{aligned}$$

Moreover, denoting $\alpha = \sup\{t \geq 0; x \in \varphi(t, \Omega)\}$ we have $-\alpha = \inf J(x)$. With $C^-(x) = \varphi((-\alpha, 0], x)$ it follows $C(x) = C^+(x) \cup C^-(x)$, $C^+(x) \cap C^-(x) = \{x\}$, and

$$\begin{aligned}
&\int_{[0,\infty)} \rho_{t,p}(x) d\lambda(t) = \int_{[0,\alpha)} \rho_{t,p}(x) d\lambda(t) \\
&= \int_{[0,\alpha)} \exp(\int_{-t}^0 p\operatorname{Re} h(\varphi(r,x)) - F'(\varphi(r,x)) dr) \rho(\varphi(-t,x)) d\lambda(t) \\
&= \int_{(-\alpha,0]} \exp(\int_t^0 p\operatorname{Re} h(\varphi(r,x)) - F'(\varphi(r,x)) dr) \rho(\varphi(t,x)) d\lambda(t) \\
&= \int_{(-\alpha,0]} \exp(\int_t^0 \frac{p\operatorname{Re} h(\varphi(r,x)) - F'(\varphi(r,x))}{F(\varphi(r,x))} \partial_1 \varphi(r,x) dr) \rho(\varphi(t,x)) d\lambda(t) \\
&= \int_{(-\alpha,0]} \exp(\int_{\varphi(t,x)}^x \frac{p\operatorname{Re} h(y) - F'(y)}{F(y)} dy) \rho(\varphi(t,x)) d\lambda(t) \\
&= \int_{(-\alpha,0]} \frac{\exp(\int_{\varphi(t,x)}^x \frac{p\operatorname{Re} h(y) - F'(y)}{F(y)} dy)}{|F(\varphi(t,x))|} \rho(\varphi(t,x)) |\partial_1 \varphi(t,x)| d\lambda(t) \\
&= \int_{C^-(x)} \frac{\exp(\int_w^x \frac{p\operatorname{Re} h(y) - F'(y)}{F(y)} dy)}{|F(w)|} \rho(w) d\lambda(w).
\end{aligned}$$

Combining these equations yields

$$\begin{aligned}
&\int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) \\
&= \int_{C(x)} \frac{\exp(\int_w^x \frac{p\operatorname{Re} h(y) - F'(y)}{F(y)} dy)}{|F(w)|} \rho(w) d\lambda(w) \\
&= \int_{C(x)} \frac{\exp(p \int_w^x \frac{\operatorname{Re} h(y)}{F(y)} dy) \exp(\log |F(w)| - \log |F(x)|)}{|F(w)|} \rho(w) d\lambda(w) \\
&= \frac{1}{|F(x)|} \int_{C(x)} \exp(p \int_w^x \frac{\operatorname{Re} h(y)}{F(y)} dy) \rho(w) d\lambda(w)
\end{aligned}$$

which proves the lemma. \square

Remark 5. The last step in the above proof shows that for $x \in \Omega \setminus \{F = 0\}$ and all $v \in C(x)$ we have for every $1 \leq p < \infty$

$$\begin{aligned}
& \int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) \\
&= \frac{1}{|F(x)|} \int_{C(x)} \exp(p \int_w^x \frac{\operatorname{Re} h(y)}{F(y)} dy) \rho(w) d\lambda(w) \\
&= \frac{|F(v)|}{|F(x)|} \exp(p \int_v^x \frac{\operatorname{Re} h(y)}{F(y)} dy) \frac{1}{|F(v)|} \int_{C(x)} \exp(p \int_x^v \frac{\operatorname{Re} h(y)}{F(y)} dy) \rho(w) d\lambda(w) \\
&= \frac{|F(v)|}{|F(x)|} \exp(p \int_v^x \frac{\operatorname{Re} h(y)}{F(y)} dy) \int_{\mathbb{R}} \rho_{t,p}(v) d\lambda(t).
\end{aligned}$$

Thus, under the hypotheses of Lemma 4 the following are equivalent for every connected component C of $\Omega \setminus \{F = 0\}$ and all $1 \leq p < \infty$.

- i) $\exists x \in C : \int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) < \infty$,
- ii) $\forall x \in C : \int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) < \infty$,
- iii) $\exists x \in C : \int_C \exp(-p \int_x^w \frac{\operatorname{Re} h(y)}{F(y)} dy) \rho(w) d\lambda(w) < \infty$,
- iv) $\forall x \in C : \int_C \exp(-p \int_x^w \frac{\operatorname{Re} h(y)}{F(y)} dy) \rho(w) d\lambda(w) < \infty$.

We have now everything at hand to prove Theorem 1.

Proof of Theorem 1. By [3, Theorem 6 and Proposition 9] $T_{F,h}$ is chaotic on $L^p_\rho(\Omega)$ if and only if $\lambda(\{F = 0\}) = 0$ as well as for every $m \in \mathbb{N}$ for which there are m different connected components C_1, \dots, C_m of $\Omega \setminus \{F = 0\}$, for λ^m -almost all choices of $(x_1, \dots, x_m) \in \prod_{j=1}^m C_j$ there is $t > 0$ such that

$$\sum_{j=1}^m \sum_{l \in \mathbb{Z}} \rho_{lt,p}(x_j) < \infty.$$

By lemma 3, this holds precisely when $\lambda(\{F = 0\}) = 0$ and when for λ -almost every $x \in \Omega \setminus \{F = 0\}$

$$\int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) < \infty.$$

Thus, applying Remark 5, Theorem 1 follows. \square .

Remark 6. a) Inspection of the proof of Theorem 1 yields the following. Under the hypothesis of Theorem 1, the following are equivalent for ρ p -admissible for F and h .

- i) $T_{F,h}$ is chaotic in $L^p_\rho(\Omega)$.
- ii) $\lambda(\{F = 0\}) = 0$ and for all $x \in \Omega \setminus \{F = 0\}$ there is $t_0 > 0$ such that $\sum_{k \in \mathbb{Z}} \rho_{kt_0,p}(x) < \infty$.
- iii) $\lambda(\{F = 0\}) = 0$ and $\sum_{k \in \mathbb{Z}} \rho_{kt_0,p}(x) < \infty$ for all $x \in \Omega \setminus \{F = 0\}$ and all $t_0 > 0$.
- iv) $\lambda(\{F = 0\}) = 0$ and $\int_{\mathbb{R}} \rho_{t,p}(x) d\lambda(t) < \infty$ for all $x \in \Omega \setminus \{F = 0\}$.
- v) $\lambda(\{F = 0\}) = 0$ and for every connected component C of $\Omega \setminus \{F = 0\}$

$$\int_C \exp(-p \int_x^w \frac{\operatorname{Re} h(y)}{F(y)} dy) \rho(w) d\lambda(w) < \infty$$

for some/all $x \in C$.

b) If $h = 0$ and if $F \in C^1(\Omega)$ is as usual then the p -admissibility of ρ does not depend on p . If moreover F' is bounded the following are then equivalent.

- i) $T_F = T_{F,0}$ is chaotic in $L^p_\rho(\Omega)$ for some/all $p \in [1, \infty)$.

ii) $\lambda(\{F = 0\}) = 0$ and for every connected component C of $\Omega \setminus \{F = 0\}$ we have

$$\int_C \rho(w) d\lambda(w) < \infty.$$

Example 7. a) Let $\Omega \in \{(0, \infty), \mathbb{R}\}$ and let $F(x) = 1$. Then Ω is forward invariant under F . Moreover, let $h \in C(\Omega)$ be such that $\operatorname{Re} h$ is bounded. It follows from the definition, that $\rho = 1$ is p -admissible for F and h for every $1 \leq p < \infty$ so that $T_{1,h}$ is a well defined C_0 -semigroup on $L^p(\Omega)$, the so-called perturbed translation semigroup. If h is bounded the generator of $T_{1,h}$ in $L^p(\Omega)$ is given by

$$A_p : W^{1,p}(\Omega) \rightarrow L^p(\Omega), A_p f(x) = f' + hf,$$

where f' denotes the distributional derivative of f (see e.g. [3, Theorem 15]).

If $\operatorname{Im} h \in L^1(0, \beta)$, resp. $\operatorname{Im} h \in L^1(-\infty, \beta)$ for all $\beta \in \Omega$ or if $\operatorname{Im} h \in L^1(\beta, \infty)$ for all $\beta \in \Omega$, by Theorem 1 this C_0 -semigroup is chaotic on $L^p(\Omega)$ if and only if

$$\int_{\Omega} \exp(-p \int_1^w \operatorname{Re} h(y) dy) d\lambda(w) < \infty.$$

b) Consider again $\Omega \in \{(0, \infty), \mathbb{R}\}$ and let $F(x) = 1$. Moreover, let ρ be p -admissible for F and $h = 0$ (which does not depend on p by Remark 6 b)). We then obtain the classical translation semigroup and Remark 6 a) gives the well-known characterizations of chaos for this semigroup due to Matsui, Yamada, and Takeo [12, 13] and deLaubenfels and Emamirad [7], respectively.

c) Consider $\Omega = (0, 1)$ and let $F(x) = -x$. Then Ω is forward invariant for F . Additionally, let $h \in C(0, 1)$ be such that $\operatorname{Re} h$ is bounded. It follows again from the definition that $\rho = 1$ is p -admissible for F and h for every $1 \leq p < \infty$. Thus, we obtain a well-defined C_0 -semigroup $T_{-id,h}$ on $L^p(0, 1)$. If h is bounded the generator of this semigroup in $L^p(\Omega)$ is given by

$$A_p : \{f \in L^p(0, 1); xf'(x) \in L^p(0, 1)\} \rightarrow L^p(\Omega), A_p f(x) = -xf'(x) + h(x)f(x),$$

where f' denotes again the distributional derivative of f (see e.g. [3, Theorem 15]).

If $x \mapsto \frac{\operatorname{Im} h(x)}{x} \in L^1(0, \beta)$ for all $\beta \in (0, 1)$ or if $x \mapsto \frac{\operatorname{Im} h(x)}{x} \in L^1(\beta, 1)$ for all $\beta \in (0, 1)$, by Theorem 1 this C_0 -semigroup is chaotic on $L^p(\Omega)$ precisely when for some $x \in (0, 1)$

$$\int_{(0,1)} \exp(-p \int_x^w \frac{\operatorname{Re} h(y)}{-y} dy) d\lambda(w) < \infty.$$

Because of

$$\exp(p \int_x^w \frac{\operatorname{Re} h(y)}{y} dy) = \left(\frac{w}{x}\right)^{p \operatorname{Re} h(0)} \exp(p \int_x^w \frac{\operatorname{Re} h(y) - \operatorname{Re} h(0)}{y} dy)$$

this generalizes a result of Dawidowicz and Poskrobko [6] who showed that in case of a real valued $h \in C[0, 1]$ for which $x \mapsto \frac{h(x) - h(0)}{x} \in L^1(0, 1)$ the above semigroup is chaotic on $L^p(0, 1)$ if and only if $h(0) > -1/p$.

d) Consider $\Omega = (0, 1)$ and $F(x) = -x^3 \sin(\frac{1}{x})$. Because we have $\lim_{x \rightarrow 0} F(x) = 0$ and $\lim_{x \rightarrow 1} F(x) \leq 0$ it follows that Ω is forward invariant under F and since F' is bounded $\rho = 1$ is p -admissible for F and $h = 0$ for every $1 \leq p < \infty$. Thus, T_F is a well-defined C_0 -semigroup on $L^p(0, 1)$. By [3, Theorem 15] its generator is

$$A_p : \{f \in L^p(0, 1); -x^3 \sin(\frac{1}{x})f'(x) \in L^p(0, 1)\} \rightarrow L^p(0, 1),$$

$$A_p f(x) = -x^2 \sin(\frac{1}{x})f'(x)$$

where f' denotes the distributional derivative of f . By Remark 6 it follows that this C_0 -semigroup is chaotic on $L^p(0, 1)$ for every $1 \leq p < \infty$.

3. WEIGHTED COMPOSITION C_0 -SEMIGROUPS ON SOBOLEV SPACES

For a bounded interval (a, b) , let $F \in C^1[a, b]$ with $F(a) = 0$ be such that (a, b) is forward invariant under F , and let $h \in W^{1,\infty}[a, b]$ be such that

- 1) $\forall x \in \{F = 0\} : h(x) = h(a) \in \mathbb{R}$,
- 2) the function $[a, b] \rightarrow \mathbb{R}, y \mapsto \frac{h(y) - h(a)}{F(y)}$ belongs to $L^\infty[a, b]$.

In [3] it is shown that under the above hypothesis the operator

$$A_p : \{f \in W^{1,p}[a, b]; Ff'' \in L^p[a, b]\} \rightarrow W^{1,p}[a, b], A_p f = Ff' + hf,$$

where the derivatives are taken in the distributional sense, is the generator of a C_0 -semigroup $S_{F,h}$ on $W^{1,p}[a, b]$ ($1 \leq p < \infty$) which is given by

$$\forall t \geq 0, f \in W^{1,p}[a, b] : S(t)f(x) = h_t(x)f(\varphi(t, x)).$$

Moreover, it is shown in [3] that this C_0 -semigroup $S_{F,h}$ is never hypercyclic on $W^{1,p}[a, b]$. In particular, $S_{F,h}$ cannot be chaotic on $W^{1,p}[a, b]$.

Because of $F(a) = 0$, the closed subspace

$$W_*^{1,p}[a, b] := \{f \in W^{1,p}[a, b]; f(a) = 0\}$$

of $W^{1,p}[a, b]$ is invariant under $S_{F,h}$ such that the restriction of $S_{F,h}$ to $W_*^{1,p}[a, b]$ defines a C_0 -semigroup on $W_*^{1,p}[a, b]$ which we denote again by $S_{F,h}$. Its generator is given by

$$A_{p,*} : \{f \in W_*^{1,p}[a, b]; Ff'' \in L^p[a, b]\} \rightarrow W_*^{1,p}[a, b], A_{p,*} f = Ff' + hf,$$

see [3]. Using Theorem 1 we derive the following characterization of chaos for $S_{F,h}$ on $W_*^{1,p}[a, b]$.

Theorem 8. *Let (a, b) be a bounded interval, $F \in C^1[a, b]$ with $F(a) = 0$ such that (a, b) is forward invariant under F . Moreover, let $h \in W^{1,\infty}[a, b]$ be such that*

- 1) $\forall x \in \{F = 0\} : h(x) = h(a) \in \mathbb{R}$,
- 2) *the function $[a, b] \rightarrow \mathbb{C}, y \mapsto \frac{h(y) - h(a)}{F(y)}$ belongs to $L^\infty[a, b]$.*

Then, for the C_0 -semigroup $S_{F,h}$ on $W_^{1,p}[a, b]$ the following are equivalent.*

- i) $S_{F,h}$ is chaotic.
- ii) $\lambda(\{F = 0\}) = 0$ and for every connected component C of $(a, b) \setminus \{F = 0\}$

$$\int_C \exp(-p \int_x^w \frac{F'(y) + h(a)}{F(y)} dy) d\lambda(w) < \infty$$

for some/all $x \in C$.

Proof. Observe that by the boundedness of F' on $[a, b]$ $\rho = 1$ is p -admissible for F and $F' + h(a)$ for any $1 \leq p < \infty$. Under the above hypothesis 1) and 2) it is shown in [3, Theorem 20 and Proposition 24] that the C_0 -semigroups $S_{F,h}$ on $W_*^{1,p}[a, b]$ and $T_{F,F'+h(a)}$ on $L^p[a, b]$ are conjugate, i.e. there is a homeomorphism $\Phi : L^p[a, b] \rightarrow W_*^{1,p}[a, b]$ such that $S_{F,h}(t) \circ \Phi = \Phi \circ T_{F,F'+h(a)}(t)$ for every $t \geq 0$. By the so-called Comparison Principle (see e.g. [9, Proposition 7.7]) it follows that $S_{F,h}$ is chaotic on $W_*^{1,p}[a, b]$ if and only if $T_{F,F'+h(a)}$ is chaotic on $L^p[a, b]$. Thus, an application of Theorem 1 proves the theorem. \square

Example 9. a) We consider $(a, b) = (0, 1)$ and $F(x) = -x$. Then, $(0, 1)$ is forward invariant under F . For every $h \in W^{1,\infty}[0, 1]$ with $h(0) \in \mathbb{R}$ and

$$[0, 1] \rightarrow \mathbb{C}, y \mapsto \frac{h(y) - h(0)}{y} \in L^\infty[0, 1]$$

the operator

$$A : \{f \in W_*^{1,p}[a, b]; xf''(x) \in L^p[a, b]\} \rightarrow W^{1,p}[a, b], Af(x) = -xf'(x) + h(x)f(x),$$

generates a C_0 -semigroup on $W_*^{1,p}[0, 1]$, $1 \leq p < \infty$. By Theorem 8 this semigroup is chaotic on $W_*^{1,p}[0, 1]$ if and only if for some $x \in (0, 1]$

$$\int_{[0,1]} \left(\frac{w}{x}\right)^{p(h(0)-1)} d\lambda(w) = \int_{[0,1]} \exp(-p \int_x^w \frac{-1+h(0)}{-y} dy) d\lambda(w) < \infty$$

which holds precisely when $p(h(0) - 1) > -1$, i.e. when $h(0) > 1 - \frac{1}{p}$ (see also [3, Theorem 27]).

b) Let again $(a, b) = (0, 1)$. We consider $F(x) = -x(1 - x)$ so that $(0, 1)$ is forward invariant under F . For each $h \in W^{1,\infty}[0, 1]$ with $h(0) = h(1) \in \mathbb{R}$ and

$$[0, 1] \rightarrow \mathbb{C}, y \mapsto \frac{h(y) - h(0)}{y(1 - y)} \in L^\infty[0, 1]$$

the operator

$$\begin{aligned} A : \{f \in W_*^{1,p}[a, b]; x(1 - x)f''(x) \in L^p[a, b]\} &\rightarrow W^{1,p}[a, b], \\ Af(x) &= -x(1 - x)f'(x) + h(x)f(x), \end{aligned}$$

generates a C_0 -semigroup on $W_*^{1,p}[0, 1]$, $1 \leq p < \infty$. Since for any $x \in (0, 1)$ the function

$$w \mapsto \exp(-p \int_x^w \frac{F'(y) - h(0)}{F(y)} dy) = w^{-p(1+h(0))} (1-w)^{-p(1-h(0))} (1-x)^{p(1-h(0))} x^{p(1+h(0))}$$

does not belongs to $L^1(0, 1)$ for any value of $h(0)$ it follows from Theorem 8 that this semigroup is not chaotic.

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